# Bifurcation and chaos in two coupled periodically driven four-well Duffing-vander Pol oscillators

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**Abstract**

Bifurcation and chaos in two coupled periodically driven four-well Duffing-vander Pol oscillators is numerically investigated as a function of nonlinear coupling(δ) and the strength of the driving force amplitude ( $f$ ). The influence of small values of coupling (δ <<1) and large values of coupling (δ >>1) is analyzed with and without external force. For large values of coupling (δ >>1), the system (1) shows completely chaotic behaviour but for small values of coupling the system(1) leads to period doubling routes to chaos, quasiperiodic orbit, reverse - period doubling bifurcation, periodic windows, chaotic orbit and intermittent behaviours for the specific set of values of the parameters of the system. We characterize periodic, quasiperiodic and chaotic orbits by constructing the bifurcation diagram, phase portrait and Poincare map.

## 1. Introduction

Recently a great deal of interest has been shown in research on coupled nonlinear oscillators under the influence of applied periodic force along with the presence of nonlinear coupling. A number of studies have been carried out to quantify the effect of nonlinear coupling and external force in coupled nonlinear oscillators and many fascinating phenomena were observed. Particularly, the occurrence of hyperchaos [1-3], migration control [4], bifurcation structure [5-7], resonance behaviour and jump phenomenon [8], basin structure [9], chaotic beats [10], vibrational and stochastic resonances [11-13], symmetry breaking bifurcations [14], double auto-resonance [15], effect of coupling on controlling and synchronization of chaos [16-18], mode interaction [19] and scaling features of multi - mode motions [20] have been reported. Analytical techniques such as multiple scale perturbation theory, Melnikov method and Painleve analysis can be used to study nonlinear resonances, homoclinic bifurcations and solution for the integribility choices of two coupled nonlinear oscillators. For example, Painleve analysis is used to identify the analytical solution of integrable choices of two coupled Duffing oscillators [21,21A] and Melnikov method is often used to study the occurrence of horseshoe chaos in two coupled Duffing oscillators [22]. Method of multiple scales is used to study the occurrence of hysteresis and nonlinear resonances in two coupled nonlinear oscillators [23-25]. Coexisting of attractors and hysteresis have been observed in two coupled Duffing oscillators [26] and in an experimental study of Colpitt's oscillator [27].

In this paper we numerically analyze the bifurcations and chaos in two coupled Duffing - vander Pol oscillators equation

$$\begin{aligned} \ddot{x} - d\,\dot{x}(1-x^2) + 2Ax + 4\alpha x^3 + 2\delta x y^2 &= f\sin\omega t \qquad (1)\\ \ddot{y} - d\,\dot{y}(1-y^2) + 2By + 4\beta y^3 + 2\delta y x^2 &= g\sin\Omega t \end{aligned}$$

Where d is the damping parameter, A and B are the natural frequencies, α and β are the coefficient of nonlinear terms, δ the coupling parameter, $f$ and g are the amplitudes of the driving force, ω and Ω the frequencies of the driving force . When the coupling parameter (δ) is set to zero, the system (1) becomes two uncoupled periodically driven Duffing - vander Pol oscillators. In the present work we wish to analyze the occurrence of bifurcations and chaos in two coupled periodically driven Duffing - vander Pol oscillators. The motivation for our interest in the system (1) is that its uncoupled version has a wide range of applications in Physics, Applied Mathematics and Biology. It is an alternative form of Bonhoeffer - vander Pol oscillator [28], driven magnetic oscillator [29] and also describes the dynamics of charge density in the plasma of a rf gas discharge. It exhibits well developed chaos in the parameter space [30,31]. Recently Gandhimathi et al. [32] studied the occurrence of chaos and hysteresis in system (1) with a nonlinear coupling and without any external sinusoidal force. In a very recent work, Kengine et al. [33] studied the experiment on bifurcation and chaos in two coupled vander Pol - Duffing oscillators.

## 2. Equilibrium points and shape of the potential.

When d = 0, $f$ = 0 and g = 0, the system (1) is the two coupled anharmonic oscillators with the potential

$$V(x,y) = A\frac{x^2}{2} + B\frac{y^2}{2} + \alpha\frac{x^4}{4} + \beta\frac{y^4}{4} + \delta\frac{x^2 y^2}{2} \qquad ----- (2)$$

The shape of the potential function (Eq.2) depends on the sign of the parameters. For A, B < 0, α, β, δ > 0 the potential has four minima. The shape of the potential can be identified by determining the maxima and minima of the function V(x,y). Figure (1) shows a typical four-well potential.

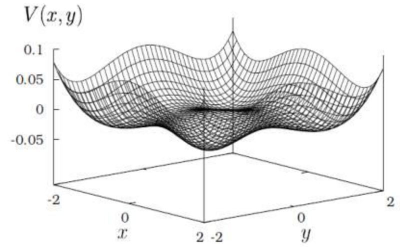


FIG. 1: A four-well potential. The values of the parameters in the system-1 are $A = -0.1$, $B = -0.15$, $\alpha = 0.1$, $\beta = 0.1$, and $\delta = 0.1$.

### 3. Bifurcations and chaos in the system (1)

In this section we show the occurrence of bifurcation and chaos in two coupled periodically driven Duffing - vander Pol oscillators with nonlinear coupling parameter (Eq.1.). System (1) involves eleven independent parameters : d, A, B, α, β, δ, *f*, g, ω, and Ω. Due to the relatively large number of parameters, the detailed influence of each parameter in the dynamics of the system (1) is not presented here. But it is of interest to analyze the influence of some parameter on the system (1). To simplify the analysis all the parameters are kept constant, except *f* ( = g), and δ. For our numerical study, we fix the parameters values as A = -0.1, B = -0.15, α = 0.1, β = 0.1 ω = Ω = 1, d = 0.2 and choose the period of the force as 2π/ω. Equation (1) is solved by the fourth-order Runge Kutta method with time step size [(2π/ω)/200] (≈ 0.031). The initial condition used is (x,y,u,v) = (0.0053,2.0141,2.2047,0). Numerical solution corresponding to first 500 drive cycles is left as transients and the next 1000 points in the Poincare map are used to identify the type of the periodic and chaotic attractors. First we analyze the occurrence of bifurcation and chaos in the unforced system (1) (ie *f* = 0, g = 0) by varying the coupling strength δ. Then we study the bifurcations and chaos in system (1) by varying the forcing amplitude (*f* = (g)) with the fixed values of δ.

### 3.1. Bifurcations and chaos in the unforced system (1)

We consider the system (1) without external force (ie *f* = 0 and g = 0). Here the control parameter is δ. Figure 2(a) shows the bifurcation phenomenon with one of the period -T attractors where δ is varied from 0 to 2. Bifurcations to period -2T, 4T orbits are found at δ = 0.654 and 0.735 respectively. Chaotic motion is observed in a small interval 0.741350 - 0.7417. At δ = 0.7417 the chaotic orbit disappears and the period -T orbit is found to occur in the interval 0.7417 < δ < 1.25. At δ = 1.25 the period -T disappears and the system completely enters into a chaotic dynamics. Magnification of part of bifurcation diagrams is shown in Fig.2(b). Figure 3 shows the phase portrait of chaotic orbit and the corresponding Poincare map for δ = 1.5.

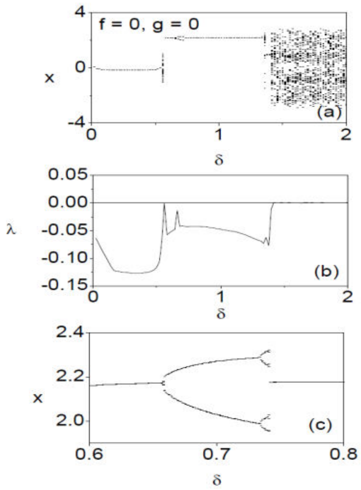


FIG. 2: (a) Bifurcation diagram obtained
bifurcation diagram of fig.2(a) The values
$-0.15, \alpha = 0.1, \beta = 0.1$, and $d = 0.2$.

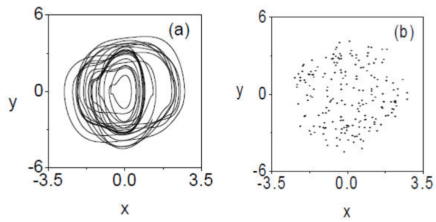


FIG. 3: Phase portrait and the corresponding Poincaré map for the chaotic orbit. The values of the parameters in the system-1 are $A = -0.1, B = -0.15, \alpha = 0.1, \beta = 0.1$, and $\delta = 1.5$.

**3.2. Bifurcations and chaos in the coupled system driven by only one periodic force**

Before studying the system (1) in the presence of both driven periodic forces in ,we consider the coupled system driven by only one force (ie $f \neq 0$ and g = 0). We show the effect of control parameters $f$ by fixing the values δ in regular region and then in a chaotic region. For δ = 0.2, $f$ = 0 and g = 0, the motion of the system is periodic with period -T. Figure 4(a) shows the bifurcation phenomenon where the values of $x$ at every period of the external drive versus the driving amplitude f is plotted for δ = 0.2 (periodic region). The corresponding maximal Lyapunov exponent diagram is shown in fig.4(b). For δ < 2.825 chaotic and periodic oscillations with period -T occurs. Quasiperiodic behaviour interrupted by periodic windows is found to occur for 2.825 < δ < 3.5. When the driving force amplitude $f$ is further increased periodic windows with higher periods exist and finally reverse bifurcation to period -1 attractor occurs. The magnification of a part of bifurcation diagram of fig.4(a) is shown in fig.4(c). For δ = 1.5, the system (1) shows completely chaotic behaviour, which is clearly evident in fig.4(d). Periodic orbits for various values of $f$ is shown in fig.5. An example of quasiperiodic orbit and chaotic orbit is shown in fig.6(a-b) and the corresponding Poincare map is shown in fig.6(c-d).

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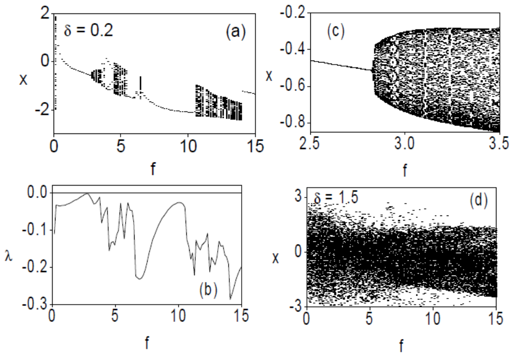


FIG. 4: Bifurcation diagrams (a) $f$ is varied from 0 to 7 for $\delta = 0.2$ (b) magnification of part of a diagram fig.4(a). (c) $f$ is varied from 0 to 10 for $\delta = 1.5$. The values of the parameters in the system-1 are $A = -0.1, B = -0.15, \alpha = 0.1, \beta = 0.1$, and $d = 0.2$.

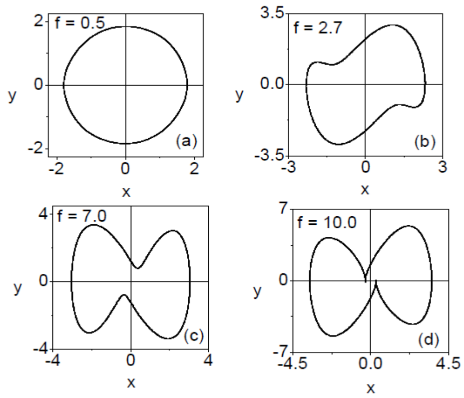


FIG. 5: Periodic orbits for various values of $f$. The values of the parameters in the system-1 are $A = -0.1, B = -0.15, \alpha = 0.1, \beta = 0.1, d = 0.2$, and $\delta = 0.2$.

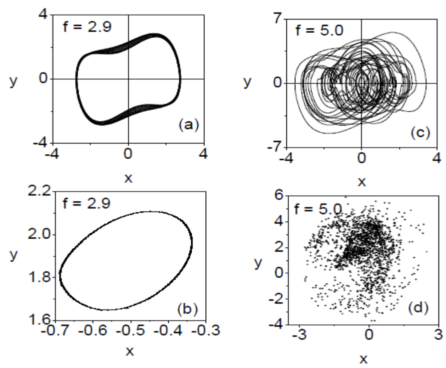


FIG. 6: Phase portraits and the corresponding Poincare maps (a-b) quasiperiodic orbit for $f = 2.7$ and $\delta = 0.2$. (c-d) Chaotic orbit for $f = 5.0$ and $\delta = 1.5$. The values of the parameters in the system-1 are $A = -0.1, B = -0.15, \alpha = 0.1, \beta = 0.1$ and $d = 0.2$.

**3.3. Bifurcations and chaos in the coupled system driven by two periodic forces**

In the previous section, we studied the occurrence of bifurcations and chaos in the coupled system driven by single periodic force. In this section, we analyze the occurrence of bifurcations and chaos in the coupled system driven by two periodic forces with the same forcing amplitudes (ie $f$ = g) . First we show the effect of the control parameter $f$ (= g) by fixing the values of δ in a regular region and then in a chaotic region. We fix the values of the parameter as A = -0.1, B = -

0.15, $\alpha = 0.1$, $\beta = 0.1$, $\omega = \Omega = 1$ and d = 0.. Figure 7(a) shows the bifurcation phenomena when $f$ (= g) is varied from 0 to 10. As $f$ (= g) is increased from zero, the coupled system starts with chaotic motion and further increase of $f$ (= g) values, the chaotic motion settles to a periodic behaviour. The effect of $\delta = 1.5$ is shown in fig.7(b).

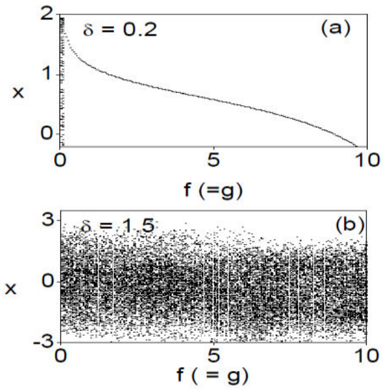


FIG. 7: Bifurcation digrams (a) for $\delta = 0.2$ and (b) for $\delta = 1.5$. The values of the parameters in the system-1 are $A = -0.1, B = -0.15, \alpha = 0.1, \beta = 0.1$ and $d = 0.2$.

**4. Conclusion**

We numerically studied the occurrence of bifurcations and chaos in two coupled periodically driven four-well Duffing - vander Pol oscillators for specific set of values of the parameters. First we have shown the occurrence of various bifurcations and chaos in the system (1) without any external force. Then we studied the bifurcations and chaos in the coupled systems driven by only one periodic force and two periodic forces by fixing the values of δ in regular and chaotic regions. Our study shows that bifurcations and chaotic phenomena are abundant in the periodically driven two coupled four - well Duffing - vander Pol oscillators in the absence of external force and with external force. In the present work we restricted our study to nonlinear coupling and the same forcing amplitudes ($f$ = g). Recently Belykh [34] analyzed the dynamical behaviours of two vander Pol - Duffing oscillators with Huygens coupling. It is of interest to investigate the occurrence of vibrational resonance, ghost - vibrational resonance and onset of chaos in the system (1) with various types of coupling and various periodic forces.

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